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Three-level coupled systems and parasupersymmetric shape invariance

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Abstract
A class of bound-state problems which represents the coupling of a three-level atom with a two-dimensional system involving two shape-invariant potentials is introduced. We consider second-order parasupersymmetric quantum-mechanical models and, using an algebraic formulation for shape-invariant potential systems, resolve the eigenvalue problem for these coupled systems considering two possible kinds for the coupling Hamiltonian (linear and nonlinear in the potential ladder operators). An application is given for a couple of shape-invariant potentials (harmonic oscillator + Morse potentials).

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1. Introduction

The concept of supersymmetry was first introduced in the early 1970s [1] in the context of a unifying treatment of bosonic and fermionic parts of the string spectrum and has become today a field in its own right with many applications to gravity [2], nuclear [3], solid state and statistical physics [4]. In particular, supersymmetry is inherent in several quantum-mechanical systems, where it allows one to establish various important properties like degeneracy of the spectrum, relations between the spectra of different Hamiltonians [5, 6], etc. Supersymmetric quantum mechanics is usually studied in the context of one-dimensional systems [4]. The partner Hamiltonians

$$\hat{H}_- = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(-)}(x) = \hbar\Omega\hat{A}^\dagger\hat{A} \qquad \text{and} \qquad \hat{H}_+ = -\frac{\hbar^2}{2M}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V^{(+)}(x) = \hbar\Omega\hat{A}\hat{A}^\dagger \tag{1}$$

can be written in terms of one-dimensional operators

$$\hat{A} \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) + \frac{\mathrm{i}}{\sqrt{2M}}\hat{p}\right) \qquad \text{and} \qquad \hat{A}^\dagger \equiv \frac{1}{\sqrt{\hbar\Omega}}\left(W(x) - \frac{\mathrm{i}}{\sqrt{2M}}\hat{p}\right), \tag{2}$$

where $\hbar\Omega$ is a constant energy scale factor, introduced so that the operators $\hat{A}$ and $\hat{A}^\dagger$ are dimensionless. The function $W(x)$ is the superpotential which is related to the partner potentials $V^{(\pm)}(x)$ via

$$V^{(\pm)}(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2M}} \frac{\mathrm{d}W(x)}{\mathrm{d}x}. \tag{3}$$

A number of such pairs of Hamiltonians $\hat{H}_\pm$ share an integrability condition called shape invariance [7]. Although not all exactly solvable problems are shape invariant [8], shape invariance, especially in its algebraic formulation [9–12], proved to be an excellent technique to investigate exactly solvable systems. Solvable models in quantum theory are so rare that they are worth studying in their own right. Even though they are simplified, they provide a clear understanding of physical phenomena involved and can be useful in controlling various approximations indispensable for treating more realistic cases. In this sense, the supersymmetric quantum mechanics allied to the shape-invariance concept represent an elegant and powerful method for obtaining closed analytic solutions for the energy eigenvalues and eigenfunctions of a set of potential system (such as Coulomb, harmonic oscillator, Morse, Eckart, Pöschl–Teller, Húlthen, etc) with application in bound states problems common in many areas of the physics [4].

In the 1950s the so-called *parastatistics* [13, 14] which generalizes the ordinary Bose and Fermi statistics introducing the concept of parabosons and parafermions was presented. In view of the fact that the supersymmetric quantum mechanics provides us with an elegant symmetry between fermions and bosons, Rubakov and Spiridonov [15] constructed the second-order parasupersymmetric quantum mechanics of one bosonic and one parafermionic degrees of freedom as a natural extension of supersymmetric quantum mechanics. Beginning with this work, parasupersymmetric quantum-mechanical systems have been exploited in many other studies [14–17], including superconformal algebra generalizations along these lines [18].

In earlier publications [19–22], we introduced a class of supersymmetric and shape-invariant coupled-channel problems which generalize the Jaynes–Cummings Hamiltonian [23], a simple model which is extensively used with success in quantum optics [24] where the radiation, represented by a harmonic oscillator, is coupled to an atom, represented by a few-level system.

In this paper, we extend to a parasupersymmetric formulation the study about coupled shape-invariant systems that we began in our previous publications [19, 22]. We consider a class of parasupersymmetric systems consisting of a shape-invariant two-dimensional system that is kept in interaction with a three-level atom. For each type of level configuration of the three-level system we consider two possible forms of coupling: one linear and one nonlinear in the generalized potential ladder operators. In this context, as a first part of our study, we present and discuss the models and, by using the algebraic formulation of shape-invariant systems presented in [9], we obtain explicit expressions for eigenvalues and eigenstates of the coupled system. Since the dynamics of this kind of coupled system is strongly dependent on the initial conditions, i.e., on the states in which the shape-invariant potential systems and the atom are prepared at the beginning, we deferred the study of the quantum dynamics of the parasupersymmetric coupled system for the second part of this study, to be presented in a forthcoming paper.

This paper is constructed as follows: In section 2, we review the basic facts of the algebraic formulation to shape invariance; in section 3, for each type of configuration of the three-level atom, we present the Hamiltonian of the coupled system, discuss the aspects of its parasupersymmetric algebra and obtain the eigenvalues and eigenfunctions for each kind of interaction (linear and nonlinear). In section 4, we apply our generalized results to a couple of shape-invariant potentials (harmonic oscillator + Morse potentials). Brief conclusions close the paper in section 5.

2. The algebraic formulation of shape invariant potential systems

The Hamiltonian $\hat{H}_-$ of equation (1) is called shape-invariant if the condition $\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1)$ is satisfied [7]. In this equation a_1 and a_2 represent parameters of the Hamiltonian. The parameter a_2 is a function of a_1 and the remainder $R(a_1)$ is independent of the dynamical variables such as position and momentum. As it is written the condition of equation (5) does not require the Hamiltonian to be one dimensional, and one does not need to choose the ansatz of equation (2). In the cases studied so far the parameters a_1 and a_2 are either related by a translation [8, 25] or a scaling [11, 26, 27]. Introducing the parameter translation operator $\hat{T} \equiv \hat{T}(a_1)$ and the similarity transformation $\hat{T}\hat{O}(a_1)\hat{T}^\dagger = \hat{O}(a_2)$ that replace a_1 with a_2 in a given operator [9, 11] and the operators

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T} \qquad \text{and} \qquad \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger\hat{A}(a_1), \tag{4}$$

the Hamiltonians of equation (1) take the forms $\hat{H}_- = \hbar\Omega\hat{\mathcal{H}}_-$ and $\hat{H}_+ = \hbar\Omega\hat{T}\hat{\mathcal{H}}_+\hat{T}^\dagger$, where $\hat{\mathcal{H}}_\pm = \hat{B}_\mp\hat{B}_\pm$. As shown in [9], with these definitions the condition of shape invariance can be written as the commutation relation

$$[\hat{B}_-, \hat{B}_+] = \hat{T}^\dagger R(a_1)\hat{T} \equiv R(a_0), \tag{5}$$

where we used the identity $R(a_n) = \hat{T}R(a_{n-1})\hat{T}^\dagger$, valid for any $n \in \mathbb{Z}$. This commutation relation suggests that $\hat{B}_-$ and $\hat{B}_+$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potentials provided that their non-commutativity with $R(a_1)$ is taken into account. The additional relations

$$R(a_n)\hat{B}_+ = \hat{B}_+R(a_{n-1}) \qquad \text{and} \qquad R(a_n)\hat{B}_- = \hat{B}_-R(a_{n+1}) \tag{6}$$

readily follow from these results. Considering that the ground state of the Hamiltonian $\hat{H}_-$ satisfies the condition $\hat{A}|0\rangle = 0 = \hat{B}_-|0\rangle$, then, using the relations above it is possible to obtain the normalized nth excited state of $\hat{\mathcal{H}}_-$

$$\hat{\mathcal{H}}_-|n\rangle = e_n|n\rangle \qquad \text{and} \qquad \hat{\mathcal{H}}_+|n\rangle = \{e_n + R(a_0)\}\,|n\rangle \tag{7}$$

from the ground state $|0\rangle$ using the relation [11]

$$|n\rangle = \hat{K}_+^n|0\rangle, \qquad \text{where} \quad \hat{K}_+ = \frac{1}{\sqrt{\hat{\mathcal{H}}_-}}\hat{B}_+. \tag{8}$$

In this case, the eigenvalues e_n are given by $e_0 = 0$ and

$$e_n = \sum_{k=1}^{n} R(a_k), \qquad \text{for} \quad n \geqslant 1. \tag{9}$$

With the results above it is possible to show that [19]

$$\hat{B}_+|n\rangle = \sqrt{e_{n+1}}|n+1\rangle, \qquad \hat{B}_-|n\rangle = \sqrt{e_{n-1} + R(a_0)}|n-1\rangle$$

and

$$\hat{T}\hat{B}_-|n+1\rangle = \sqrt{e_{n+1}}\hat{T}|n\rangle. \tag{10}$$

3. Notation and description of three-level systems coupled with shape-invariant potentials

3.1. The free atom Hamiltonian

In this study, we treat three interacting systems consisting of a single three-level atom or molecule simultaneously interacting with two shape-invariant potentials systems. If we consider that the eigenstates of the Hamiltonian $\hat{H}_{\mathrm{A}}$ of a non-interacting three-level atom $\hat{H}_{\mathrm{A}}|j\rangle_{\mathrm{A}} = \hbar\omega_j|j\rangle_{\mathrm{A}}$, with $j = 1, 2, 3$, form a basis then we must have

$$ {}_{\mathrm{A}}\langle j|k\rangle_{\mathrm{A}} = \delta_{jk} \qquad \text{and} \qquad \sum_{j=1}^{3} |j\rangle_{\mathrm{AA}}\langle j| = \hat{1\!\!1}. \tag{11} $$

Therefore using the orthonormality and the completeness relations (11) together with the eigenvalue equation $\hat{H}_{\mathrm{A}}|j\rangle_{\mathrm{A}} = \hbar\omega_j|j\rangle_{\mathrm{A}}$, it is possible to write the free atom Hamiltonian in the form

$$ \hat{H}_{\mathrm{A}} = \hat{1\!\!1}\hat{H}_{\mathrm{A}}\hat{1\!\!1} = \hbar\sum_{j,k=1}^{3} \omega_k|j\rangle_{\mathrm{AA}}\langle j|k\rangle_{\mathrm{AA}}\langle k| = \hbar\sum_{j=1}^{3}\omega_j\hat{\sigma}_{jj}, \tag{12} $$

where the projection operator $\hat{\sigma}_{jj} \equiv |j\rangle_{\mathrm{AA}}\langle j|$ describes the population of the level j which energy is $\hbar\omega_j$. By assuming a three-dimension spinor representation χ_j for the eigenstates of the atom

$$ \chi_1 \equiv \langle\chi|1\rangle_{\mathrm{A}} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad \chi_2 \equiv \langle\chi|2\rangle_{\mathrm{A}} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad \chi_3 \equiv \langle\chi|3\rangle_{\mathrm{A}} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{13} $$

it is straightforward to verify that in such representation the Hamiltonian (12) has the matrix form

$$ \mathbf{\hat{H}}_{\mathrm{A}} = \hbar\sum_{j=1}^{3}\omega_j\chi_j\chi_j^{\dagger} = \hbar\begin{bmatrix}\omega_1 & 0 & 0\\ 0 & \omega_2 & 0\\ 0 & 0 & \omega_3\end{bmatrix}. \tag{14} $$

3.2. The parasupersymmetric model for Ξ configuration

The case of the two-level system leads to ordinary supersymmetric quantum mechanics [5], while in the case of non-degenerate multilevel system (three or more levels) correspond to parasupersymmetric quantum mechanics [15, 16]. In the case of a three-level system there are three distinct energy level configurations known as Ξ or cascade configuration, Λ or Raman configuration and V configuration [28]. These configurations and their transitions are illustrated in figure 1. The total Hamiltonian describing a three-level system and two shape-invariant potentials coupled may be written as $\hat{H}_{\mathrm{T}}^{(\mathrm{X})} = \hat{H}_{\mathrm{A}} + \hat{H}_{\mathrm{P}}^{(\mathrm{X})} + \hat{H}_{\xi}^{(\mathrm{X})}$, where $\hat{H}_{\mathrm{A}}$ is the free atom Hamiltonian (12), $\hat{H}_{\mathrm{P}}^{(\mathrm{X})}$ is the Hamiltonian related with the shape-invariant potentials system and $\hat{H}_{\xi}^{(\mathrm{X})}$ is the atom–potentials interaction Hamiltonian. In a parasupersymmetric study the Hamiltonian $\hat{H}_{\mathrm{X}} = \hat{H}_{\mathrm{P}}^{(\mathrm{X})} + \hat{H}_{\xi}^{(\mathrm{X})}$ depends on the possible types of level configurations ($\mathrm{X} = \Xi, \Lambda, \mathrm{V}$) and can be expressed in terms of atomic-transition operators $\hat{\sigma}_{jk} \equiv |j\rangle_{\mathrm{AA}}\langle k|$ from the level k to the level j (with $j, k = 1, 2, 3$ and $j \neq k$) and the ladder operators $\hat{B}_{\pm}^{(k)}$ $(k = 1, 2)$ of the shape-invariant coupling potentials. The collective transition $\hat{\sigma}_{jk}$ and projection $\hat{\sigma}_{jj}$ operators obey the relations

$$ \hat{\sigma}_{jk}\hat{\sigma}_{rs} = \delta_{kr}\hat{\sigma}_{js} \qquad \text{and} \qquad \sum_{j=1}^{3}\hat{\sigma}_{jj} = \hat{1\!\!1}, \tag{15} $$

together with the commutation and anticommutation relations

$$ [\hat{\sigma}_{jk}, \hat{\sigma}_{rs}] = \delta_{kr}\hat{\sigma}_{js} - \delta_{js}\hat{\sigma}_{rk}, \qquad \{\hat{\sigma}_{jk}, \hat{\sigma}_{rs}\} = \delta_{kr}\hat{\sigma}_{js} + \delta_{js}\hat{\sigma}_{rk} \tag{16} $$

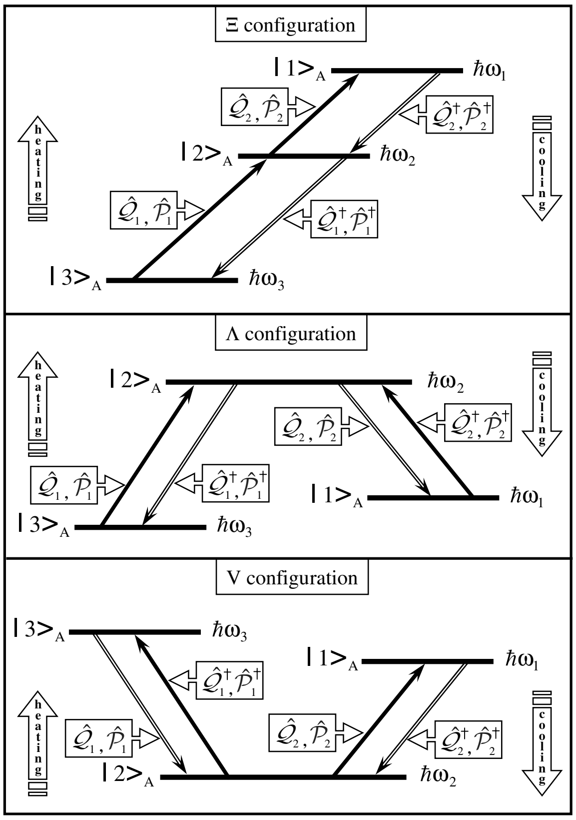

Figure 1. The energy level diagrams for the three possible configurations (Ξ, Λ and V) of a three-level system. The *heating* and *cooling* process of the coupled system, produced with the action of the parasupercharge operators $\hat{\mathcal{Q}}_{1,2}$ and $\hat{\mathcal{Q}}_{1,2}^\dagger$ in the case of a linear interaction or $\hat{\mathcal{P}}_{1,2}$ and $\hat{\mathcal{P}}_{1,2}^\dagger$ when the interaction is nonlinear, are represented schematically.

characteristic of the generators of the group $SU(3)$. A useful realization of the parafermionic operators is in terms of 3×3 matrices [15, 17]. Therefore, by assuming the three-dimension spinor representation (13) of the eigenstates of the atomic system the transition operators must be represented by the matrices

$$\hat{\boldsymbol{\sigma}}_{12} = \chi_1 \chi_2^\dagger = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\sigma}}_{23} = \chi_2 \chi_3^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\sigma}}_{13} = \chi_1 \chi_3^\dagger = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{17}$$

and

$$\hat{\boldsymbol{\sigma}}_{21} = \chi_2 \chi_1^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\sigma}}_{32} = \chi_3 \chi_2^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \hat{\boldsymbol{\sigma}}_{31} = \chi_3 \chi_1^\dagger = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{18}$$

where $\hat{\boldsymbol{\sigma}}_{kj} = \hat{\boldsymbol{\sigma}}_{jk}^\dagger$. Actually we can identify the transition matrices $\hat{\boldsymbol{\sigma}}_{jk} (j \neq k)$ with the matrix representation of the step operators (root vectors) $\hat{T}_\pm$, $\hat{U}_\pm$, and $\hat{V}_\pm$ used in the construction of the T-, U-, and V-spins $su(2)$ subalgebras. Note that with the atomic-transition-projection matrices $\hat{\boldsymbol{\sigma}}_{jk}$ it is possible to obtain the Hermitian traceless Gell–Mann matrices $\hat{\boldsymbol{\lambda}}_{jk}$. The Gell–Mann matrices [29], that generalize the 2×2 Pauli matrices $\hat{\boldsymbol{\sigma}}_i (i = 1, 2, 3)$, form a useful representation of $SU(3)$ group generators for computations in the quark model and in quantum chromodynamics.

3.2.1. The coupling potentials and the interaction Hamiltonians. We construct the Hamiltonian $\hat{H}_{\mathrm{X}}$ based in the requirements imposed by the second-order multidimensional parasuperalgebra [30] and by the dipole and rotating wave approximations, widely used in quantum optical models. For each type of configuration, we also consider two possible forms of interaction which correspond to the shape-invariant generalization of the ordinary or usual and intensity dependent or nonlinear interaction forms [31] also used in quantum optics. The usual interaction Hamiltonian is linear in the atom and the coupling potential operators, while the intensity-dependent one has that expression nonlinear in the coupling potential operators. The intensity-dependent interaction makes the enhancement of certain quantum effects possible [32] that would be otherwise difficult to note within the realm of the ordinary interaction model. We assume that each shape-invariant potential interacts with only one couple of levels in such a way that direct transitions are allowed between atomic levels 1 and 2 and between levels 2 and 3, and forbidden between levels 1 and 3. With these assumptions, we introduce the parasupersymmetric coupling potentials Hamiltonian in a Ξ-type of configuration as

$$\hat{H}_{\mathrm{P}}^{(\Xi)} = \hbar\Omega\big\{\big(\hat{A}_1\hat{A}_1^\dagger + \hat{A}_2\hat{A}_2^\dagger\big)\hat{\sigma}_{11} + \big(\hat{A}_1\hat{A}_1^\dagger + \hat{A}_2^\dagger\hat{A}_2\big)\hat{\sigma}_{22} + \big(\hat{A}_1^\dagger\hat{A}_1 + \hat{A}_2^\dagger\hat{A}_2\big)\hat{\sigma}_{33}\big\}, \tag{19}$$

where the operators $\hat{A}_1$ and $\hat{A}_2$ are related, respectively, to the two couples of potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$ and satisfy the shape-invariance condition (5). For a usual interaction, specified when $\xi = \mathrm{U}$, the interaction Hamiltonian is assumed to have the form

$$\hat{H}_{\mathrm{U}}^{(\Xi)} = \hbar g\big\{\big(\hat{A}_1\hat{\sigma}_{23} + \hat{A}_1^\dagger\hat{\sigma}_{32}\big) + \big(\hat{A}_2\hat{\sigma}_{12} + \hat{A}_2^\dagger\hat{\sigma}_{21}\big)\big\}, \tag{20}$$

while for a nonlinear interaction, specified when $\xi = \mathrm{N}$, it is assumed to be

$$\hat{H}_{\mathrm{N}}^{(\Xi)} = \hbar g\big\{\big(\hat{A}_1\sqrt{\hat{N}_1}\hat{\sigma}_{23} + \sqrt{\hat{N}_1}\hat{A}_1^\dagger\hat{\sigma}_{32}\big) + \big(\hat{A}_2\sqrt{\hat{N}_2}\hat{\sigma}_{12} + \sqrt{\hat{N}_2}\hat{A}_2^\dagger\hat{\sigma}_{21}\big)\big\}, \tag{21}$$

where g is the real coupling constant strength and $\hat{N}_k = \hat{A}_k^\dagger\hat{A}_k$ with $k = 1, 2$. Note that if we take the Hamiltonian $\hat{H}_\xi^{(\Xi)}$ for the harmonic oscillator potential, the simplest shape-invariant potential, we have the usual and intensity-dependent versions of a three-level atom interacting resonantly with a two-mode cavity field.

The algebraic formulation for shape-invariant systems presented in section 2 can be applied in the Hamiltonian $\hat{H}_\Xi$ by using the $\hat{B}_\pm^{(1,2)}$ operators defined by equations (4) with the introduction of the parameter translation operators $\hat{T}_1 \equiv \hat{T}_1\big(a_1^{(1)}\big)$ and $\hat{T}_2 \equiv \hat{T}_2\big(a_1^{(2)}\big)$ for each shape-invariant potential. Noting that the commutation relations $\big[\hat{B}_\pm^{(k)}, \hat{\sigma}_{ij}\big] = 0$, $\big[\hat{B}_\mp^{(k)}, \hat{B}_\pm^{(j)}\big] = \pm R_k(a_0^{(k)})\delta_{kj}$ and $\big[\hat{B}_\pm^{(k)}, \hat{B}_\pm^{(j)}\big] = 0$ are satisfied, where $R_1\big(a_n^{(1)}\big)$ and $R_2\big(a_n^{(2)}\big)$ are the remainders related with the potentials $V_1^{(\pm)}(x)$ and $V_2^{(\pm)}(y)$, respectively, the final result can be written as $\hat{H}_\Xi = \hat{\mathcal{T}}_\Xi\hat{h}_\Xi\hat{\mathcal{T}}_\Xi^\dagger$ if we define the parameter translation inclusive operator $\hat{\mathcal{T}}_\Xi = \hat{T}_1\hat{T}_2\hat{\sigma}_{11} + \hat{T}_1\hat{\sigma}_{22} + \hat{\sigma}_{33}$ and decompose the Hamiltonian $\hat{h}_\Xi$ in $\hat{h}_\Xi = \hat{h}_{\mathrm{P}}^{(\Xi)} + \hat{h}_\xi^{(\Xi)}$ with

$$\hat{h}_{\mathrm{P}}^{(\Xi)} = \hbar\Omega\big\{\big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_+^{(2)}\big)\hat{\sigma}_{11} + \big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{22} + \big(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{33}\big\}, \tag{22}$$

where $\hat{\mathcal{H}}_\pm^{(k)} = \hat{B}_\mp^{(k)}\hat{B}_\pm^{(k)}$, with $k = 1, 2$. The interaction Hamiltonian in the shape-invariant algebraic formulation for the two cases considered in our study is given by

$$\hat{h}_{\mathrm{U}}^{(\Xi)} = \hbar g\big(\hat{B}_-^{(1)}\hat{\sigma}_{23} + \hat{B}_+^{(1)}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\hat{\sigma}_{12} + \hat{B}_+^{(2)}\hat{\sigma}_{21}\big) \tag{23}$$

$$\hat{h}_{\mathrm{N}}^{(\Xi)} = \hbar g\Big(\hat{B}_-^{(1)}\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{\sigma}_{23} + \sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{B}_+^{(1)}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{\sigma}_{12} + \sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{B}_+^{(2)}\hat{\sigma}_{21}\Big). \tag{24}$$

In this case, we used the fact that $\hat{N}_k = \hat{A}_k^\dagger\hat{A}_k = \hat{\mathcal{H}}_-^{(k)}$ and the unitary property $\hat{T}_k^\dagger\hat{T}_k = \hat{T}_k\hat{T}_k^\dagger = \hat{\mathbb{1}}$, with $k = 1, 2$.

3.2.2. Parasupersymmetric algebra. Parasupersymmetry transformations of second order are operations that mix bosonic degrees of freedom with parafermionic variables of order $p = 2$. The generators of such transformations satisfy $(p + 1)$-linear relations that define structures called parasuperalgebras. The standard supersymmetry transformations correspond to the case where $p = 1$. In the case of our coupled system defining the parasupercharge operators as

$$\hat{\mathcal{Q}}_1 = \hat{B}_-^{(1)}\hat{\sigma}_{23}, \quad \hat{\mathcal{Q}}_1^\dagger = \hat{B}_+^{(1)}\hat{\sigma}_{32} \qquad \text{and} \qquad \hat{\mathcal{Q}}_2 = \hat{B}_-^{(2)}\hat{\sigma}_{12}, \quad \hat{\mathcal{Q}}_2^\dagger = \hat{B}_+^{(2)}\hat{\sigma}_{21}, \tag{25}$$

it is possible to rewrite the Hamiltonian in (22) in terms of the anticommutator of these operators as

$$\hat{h}_{\mathrm{P}}^{(\Xi)} = \hbar\Omega\big(\{\hat{\mathcal{Q}}_1, \hat{\mathcal{Q}}_1^\dagger\} + \{\hat{\mathcal{Q}}_2, \hat{\mathcal{Q}}_2^\dagger\} + \hat{\mathcal{Q}}_{12}\hat{\mathcal{Q}}_{12}^\dagger\big), \tag{26}$$

where $\hat{\mathcal{Q}}_{12} = \hat{B}_-^{(1)}\hat{\sigma}_{13} + \hat{B}_+^{(2)}\hat{\sigma}_{31}$ and $\hat{\mathcal{Q}}_{12}^\dagger = \hat{B}_+^{(1)}\hat{\sigma}_{31} + \hat{B}_-^{(2)}\hat{\sigma}_{13}$. Using definitions (25) and expression (26) it is easy to verify the relations

$$\big[\hat{\mathcal{Q}}_k, \hat{h}_{\mathrm{P}}^{(\Xi)}\big] = \big[\hat{\mathcal{Q}}_k^\dagger, \hat{h}_{\mathrm{P}}^{(\Xi)}\big] = 0 \qquad \text{and} \qquad \hat{\mathcal{Q}}_k^2 = \big(\hat{\mathcal{Q}}_k^\dagger\big)^2 = 0 \qquad \text{with} \qquad k = 1, 2, \tag{27}$$

$$\hat{\mathcal{Q}}_1\hat{h}_{\mathrm{P}}^{(\Xi)} = \hbar\Omega\big(\hat{\mathcal{Q}}_1\hat{\mathcal{Q}}_1^\dagger + \hat{\mathcal{Q}}_2^\dagger\hat{\mathcal{Q}}_2\big)\hat{\mathcal{Q}}_1, \qquad \hat{h}_{\mathrm{P}}^{(\Xi)}\hat{\mathcal{Q}}_1^\dagger = \hbar\Omega\hat{\mathcal{Q}}_1^\dagger\big(\hat{\mathcal{Q}}_1\hat{\mathcal{Q}}_1^\dagger + \hat{\mathcal{Q}}_2^\dagger\hat{\mathcal{Q}}_2\big), \tag{28}$$

$$\hat{\mathcal{Q}}_2\hat{h}_{\mathrm{P}}^{(\Xi)} = \hbar\Omega\hat{\mathcal{Q}}_2\big(\hat{\mathcal{Q}}_1\hat{\mathcal{Q}}_1^\dagger + \hat{\mathcal{Q}}_2^\dagger\hat{\mathcal{Q}}_2\big), \qquad \hat{h}_{\mathrm{P}}^{(\Xi)}\hat{\mathcal{Q}}_2^\dagger = \hbar\Omega\big(\hat{\mathcal{Q}}_1\hat{\mathcal{Q}}_1^\dagger + \hat{\mathcal{Q}}_2^\dagger\hat{\mathcal{Q}}_2\big)\hat{\mathcal{Q}}_2^\dagger. \tag{29}$$

Other triple products of the parasupercharge operators $\hat{\mathcal{Q}}_k$ and $\hat{\mathcal{Q}}_k^\dagger$ vanish. The set of relations (26), (27), (28) and (29) characterizes a two-dimensional generalization of the parasupersymmetric algebra with the operators $\hat{\mathcal{Q}}_k$ and $\hat{\mathcal{Q}}_k^\dagger$ as its generators [30]. On the other hand, the operators $\hat{\mathcal{Q}}_k$ and $\hat{\mathcal{Q}}_k^\dagger$ which change bosonic degrees of freedom into parafermionic ones and vice versa are related to the parasupercharge operators $\hat{Q}$ and $\hat{Q}^\dagger$ by $\hat{Q} = \frac{1}{\sqrt{2}}\big(\hat{\mathcal{Q}}_1^\dagger + \mathrm{i}\hat{\mathcal{Q}}_2^\dagger\big)$ and $\hat{Q}^\dagger = \frac{1}{\sqrt{2}}(\hat{\mathcal{Q}}_1 - \mathrm{i}\hat{\mathcal{Q}}_2)$ introduced in the two-dimensional parasupersymmetric generalization of the Witten supersymmetric Hamiltonian [33]. Note that $\hat{Q}$ and $\hat{Q}^\dagger$ obey the property $\hat{Q}^3 = (\hat{Q}^\dagger)^3 = 0$ required for parasupercharge operators of second order $\{\hat{Q}^{p+1} = (\hat{Q}^\dagger)^{p+1} = 0, \text{with} p = 2\}$.

Since in the usual interaction case the Hamiltonian in (23) can be written as

$$\hat{h}_{\mathrm{U}}^{(\Xi)} = \hbar g\big(\hat{\mathcal{Q}}_1 + \hat{\mathcal{Q}}_1^\dagger + \hat{\mathcal{Q}}_2 + \hat{\mathcal{Q}}_2^\dagger\big), \tag{30}$$

then the commutation relations (27) imply that $\big[\hat{h}_{\mathrm{P}}^{(\Xi)}, \hat{h}_{\mathrm{U}}^{(\Xi)}\big] = 0$. In the nonlinear interaction case, the Hamiltonian in (24) can be written as

$$\hat{h}_{\mathrm{N}}^{(\Xi)} = \hbar g\big(\hat{\mathcal{P}}_1 + \hat{\mathcal{P}}_1^\dagger + \hat{\mathcal{P}}_2 + \hat{\mathcal{P}}_2^\dagger\big), \tag{31}$$

where the new operators defined as $\hat{\mathcal{P}}_k = \hat{\mathcal{Q}}_k\sqrt{\hat{\mathcal{H}}_-^{(k)}}$ and $\hat{\mathcal{P}}_k^\dagger = \sqrt{\hat{\mathcal{H}}_-^{(k)}}\hat{\mathcal{Q}}_k^\dagger$ satisfy the commutation relations $\big[\hat{\mathcal{P}}_k, \hat{h}_{\mathrm{P}}^{(\Xi)}\big] = \big[\hat{\mathcal{P}}_k^\dagger, \hat{h}_{\mathrm{P}}^{(\Xi)}\big] = 0$ that imply in the compatibility condition $\big[\hat{h}_{\mathrm{P}}^{(\Xi)}, \hat{h}_{\mathrm{N}}^{(\Xi)}\big] = 0$ of the two Hamiltonians. At this point, we would like to point out that for a non-interacting system ($g = 0$) the operators $\hat{\mathcal{Q}}_k$ and $\hat{\mathcal{P}}_k$ and their Hermitian adjoint operators are conserved quantities. To conclude this section, we can say that the parasupercharge operator

$\hat{\mathcal{Q}}_k$ and its Hermitian adjoint operator $\hat{\mathcal{Q}}_k^\dagger$ in the usual interaction Hamiltonian $\hat{h}_{\mathrm{U}}^{(\Xi)}$ and the operators $\hat{\mathcal{P}}_k$ and $\hat{\mathcal{P}}_k^\dagger$ in the nonlinear interaction Hamiltonian $\hat{h}_{\mathrm{N}}^{(\Xi)}$ are responsible, respectively, for the *heating* and *cooling* process of the coupled system, schematically illustrated in figure 1.

3.2.3. Eigenstates and eigenvalues. Taking into account the compatibility of $\hat{h}_{\mathrm{P}}^{(\Xi)}$ and $\hat{h}_{\xi}^{(\Xi)}$ and using their relations with $\hat{H}_{\mathrm{P}}^{(\Xi)}$ and $\hat{H}_{\xi}^{(\Xi)}$, we find that $\left[\hat{H}_{\mathrm{P}}^{(\Xi)}, \hat{H}_{\xi}^{(\Xi)}\right] = \hat{\mathcal{T}}_\Xi\left[\hat{h}_{\mathrm{P}}^{(\Xi)}, \hat{h}_{\xi}^{(\Xi)}\right]\hat{\mathcal{T}}_\Xi^\dagger = 0$ which proves that the Hamiltonians $\hat{H}_{\mathrm{P}}^{(\Xi)}$ and $\hat{H}_{\xi}^{(\Xi)}$ share a common set of eigenstates. In this case to resolve the eigenstates equation $\hat{H}_{\xi}^{(\Xi)}\left|\Psi_\alpha^{(\xi)}\right\rangle = \mathcal{E}_\alpha^{(\xi)}\left|\Psi_\alpha^{(\xi)}\right\rangle$, we introduce the dressed states

$$\left|\Psi_\alpha^{(\xi)}\right\rangle = \hat{\mathcal{T}}_\Xi\left\{C_{1\alpha}^{(\xi)}|n_1\rangle_1|m_1\rangle_2|1\rangle_{\mathrm{A}} + C_{2\alpha}^{(\xi)}|n_2\rangle_1|m_2\rangle_2|2\rangle_{\mathrm{A}} + C_{3\alpha}^{(\xi)}|n_3\rangle_1|m_3\rangle_2|3\rangle_{\mathrm{A}}\right\}, \tag{32}$$

where $C_{j\alpha}^{(\xi)} \equiv C_{j\alpha}^{(\xi)}\left[R_1\left(a_1^{(1)}\right), R_1\left(a_2^{(1)}\right), \ldots; R_2\left(a_1^{(2)}\right), R_2\left(a_2^{(2)}\right), \ldots\right]$ are expansion coefficients which can depend on the remainders $R_k\left(a_n^{(k)}\right)$. The states $|\nu\rangle_k$, with $k = 1, 2$, are the eigenstates (8) of the coupling potential Hamiltonians $\hat{\mathcal{H}}_-^{(k)}$ with eigenvalues $e_0^{(k)} = 0$ when $\nu = 0$ and

$$e_\nu^{(k)} = \sum_{j=1}^{\nu} R_k\left(a_j^{(k)}\right), \qquad \text{for} \quad \nu \geqslant 1. \tag{33}$$

For the $\xi = \mathrm{U}$ interaction case, using the $\hat{\mathcal{T}}_\Xi$ operator expression, equations (23) and (32) in the eigenvalue equation and taking into account the operators action (10) on the eigenstates $|\nu\rangle_k$, we obtain the system of equations

$$\begin{cases} \hbar g\sqrt{e_{m_2}^{(2)}}\left(\hat{T}_1 C_{2\alpha}^{(\mathrm{U})}\hat{T}_1^\dagger\right)\hat{T}_1\hat{T}_2|n_2\rangle_1|m_2-1\rangle_2 = \mathcal{E}_\alpha^{(\mathrm{U})}\left(\hat{T}_1\hat{T}_2 C_{1\alpha}^{(\mathrm{U})}\hat{T}_2^\dagger\hat{T}_1^\dagger\right)\hat{T}_1\hat{T}_2|n_1\rangle_1|m_1\rangle_2 \\ \hbar g\left\{\sqrt{e_{m_1+1}^{(2)}}\left(\hat{T}_1\hat{T}_2 C_{1\alpha}^{(\mathrm{U})}\hat{T}_2^\dagger\hat{T}_1^\dagger\right)\hat{T}_1|n_1\rangle_1|m_1+1\rangle_2 + \sqrt{e_{n_3}^{(1)}}C_{3\alpha}^{(\mathrm{U})}\hat{T}_1|n_3-1\rangle_1|m_3\rangle_2\right\} \\ \quad = \mathcal{E}_\alpha^{(\mathrm{U})}\left(\hat{T}_1 C_{2\alpha}^{(\mathrm{U})}\hat{T}_1^\dagger\right)\hat{T}_1|n_2\rangle_1|m_2\rangle_2 \\ \hbar g\sqrt{e_{n_2+1}^{(1)}}\left(\hat{T}_1 C_{2\alpha}^{(\mathrm{U})}\hat{T}_1^\dagger\right)|n_2+1\rangle_1|m_2\rangle_2 = \mathcal{E}_\alpha^{(\mathrm{U})}C_{3\alpha}^{(\mathrm{U})}|n_3\rangle_1|m_3\rangle_2. \end{cases} \tag{34}$$

Comparing the coupling potential eigenstates $|\nu\rangle_k$ in the three equations of (34), we conclude that we must have

$$n_1 = n_2 = n, \qquad n_3 = n+1, \qquad m_1 = m, \qquad m_2 = m_3 = m+1, \tag{35}$$

and using these back in (34) we obtain a system of three equation for the coefficients $C_j^{(\mathrm{U})}$ and the eigenvalues $\mathcal{E}_\alpha^{(\mathrm{U})}$ that can be written in the matrix form $\mathbf{MC} = \mathbf{0}$, where

$$\mathbf{M} \equiv \begin{bmatrix} -\mathcal{E}_\alpha^{(\mathrm{U})} & \hbar g\sqrt{e_{m+1}^{(2)}} & 0 \\ \hbar g\sqrt{e_{m+1}^{(2)}} & -\mathcal{E}_\alpha^{(\mathrm{U})} & \hbar g\sqrt{e_{n+1}^{(1)}} \\ 0 & \hbar g\sqrt{e_{n+1}^{(1)}} & -\mathcal{E}_\alpha^{(\mathrm{U})} \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} \equiv \begin{bmatrix} \hat{T}_1\hat{T}_2 C_{1\alpha}^{(\mathrm{U})}\hat{T}_2^\dagger\hat{T}_1^\dagger \\ \hat{T}_1 C_{2\alpha}^{(\mathrm{U})}\hat{T}_1^\dagger \\ C_{3\alpha}^{(\mathrm{U})} \end{bmatrix}. \tag{36}$$

The secular equation $\det\mathbf{M} = 0$ gives the algebraic equation $\left(\mathcal{E}_\alpha^{(\mathrm{U})}\right)^3 - (\hbar g)^2\left\{e_{n+1}^{(1)} + e_{m+1}^{(2)}\right\}\mathcal{E}_\alpha^{(\mathrm{U})} = 0$ the roots of which

$$\mathcal{E}_{1nm}^{(\mathrm{U})} = +\hbar g\sqrt{e_{n+1}^{(1)} + e_{m+1}^{(2)}}, \qquad \mathcal{E}_{2nm}^{(\mathrm{U})} = 0, \qquad \text{and} \qquad \mathcal{E}_{3nm}^{(\mathrm{U})} = -\hbar g\sqrt{e_{n+1}^{(1)} + e_{m+1}^{(2)}} \tag{37}$$

are the eigenvalues of $\hat{H}_{\mathrm{U}}^{(\Xi)}$. The eigenstates associated with the eigenvalues $\mathcal{E}_{jnm}^{(\mathrm{U})}$, with $j = 1, 2, 3$, are given by

$$\left|\Psi_{jnm}^{(\mathrm{U})}\right\rangle = \hat{\mathcal{T}}_{\Xi}\left\{C_{1jnm}^{(\mathrm{U})}|n\rangle_1|m\rangle_2|1\rangle_{\mathrm{A}} + C_{2jnm}^{(\mathrm{U})}|n\rangle_1|m+1\rangle_2|2\rangle_{\mathrm{A}} + C_{3jnm}^{(\mathrm{U})}|n+1\rangle_1|m+1\rangle_2|3\rangle_{\mathrm{A}}\right\}, \quad (38)$$

with the coefficients satisfying the conditions

$$\left\langle\Psi_{jnm}^{(\mathrm{U})}\middle|\Psi_{jnm}^{(\mathrm{U})}\right\rangle = \left[C_{1jnm}^{(\mathrm{U})}\right]^2 + \left[C_{2jnm}^{(\mathrm{U})}\right]^2 + \left[C_{3jnm}^{(\mathrm{U})}\right]^2 = 1 \quad (39)$$

$$\hbar g\left(\hat{T}_1 C_{2jnm}^{(\mathrm{U})}\hat{T}_1^{\dagger}\right) = \frac{\mathcal{E}_{jnm}^{(\mathrm{U})}}{\sqrt{e_{m+1}^{(2)}}}\left(\hat{T}_1\hat{T}_2 C_{1jnm}^{(\mathrm{U})}\hat{T}_2^{\dagger}\hat{T}_1^{\dagger}\right) = \frac{\mathcal{E}_{jnm}^{(\mathrm{U})}}{\sqrt{e_{n+1}^{(1)}}} C_{3jnm}^{(\mathrm{U})}. \quad (40)$$

Following the same procedure for the $\xi = \mathrm{N}$ interaction case and using equations (24) and (32) in the eigenvalue equation $\hat{H}_{\mathrm{N}}^{(\Xi)}\left|\Psi_{\alpha}^{(\mathrm{N})}\right\rangle = \mathcal{E}_{\alpha}^{(\mathrm{N})}\left|\Psi_{\alpha}^{(\mathrm{N})}\right\rangle$ and taking into account the operators action (7) and (10) on the $|\nu\rangle_k$ coupling potential eigenstates, we obtain a system of equations similar to (34) but with the coupling potential eigenvalue factors $\sqrt{e_{\nu}^{(k)}}$ replaced by $e_{\nu}^{(k)}$. In these circumstances it is trivial to show that the eigenvalues are given by

$$\mathcal{E}_{1nm}^{(\mathrm{N})} = +\hbar g\sqrt{\left\{e_{n+1}^{(1)}\right\}^2 + \left\{e_{m+1}^{(2)}\right\}^2}, \qquad \mathcal{E}_{2nm}^{(\mathrm{N})} = 0, \qquad \text{and} \qquad \mathcal{E}_{3nm}^{(\mathrm{N})} = -\hbar g\sqrt{\left\{e_{n+1}^{(1)}\right\}^2 + \left\{e_{m+1}^{(2)}\right\}^2} \quad (41)$$

with the associated eigenstates obtained by equation (38) where the coefficients $C_{ijnm}^{(\mathrm{N})}$ satisfy similar conditions presented in equations (39) and (40) but with the replacement $\sqrt{e_{\nu}^{(k)}} \to e_{\nu}^{(k)}$ of the coupling potential eigenvalue factors in (40).

Note that the eigenstates $\left|\Psi_{jnm}^{(\mathrm{U})}\right\rangle$ and $\left|\Psi_{jnm}^{(\mathrm{N})}\right\rangle$ for the two kinds of the interaction Hamiltonians $\hat{H}_{\mathrm{U}}^{(\Xi)}$ and $\hat{H}_{\mathrm{N}}^{(\Xi)}$ have the same composition of coupling potential eigenstates $|\nu\rangle_k$ but different expansion coefficients $C_{ijnm}^{(\xi)}$. However, using equations (7), (22) and (38) it is straightforward to verify that both of them are eigenstates of the Hamiltonian $\hat{H}_{\mathrm{P}}^{(\Xi)}$ with the same set of eigenvalues

$$E_{nm} = \hbar\Omega\left\{e_{n+1}^{(1)} + e_{m+1}^{(2)}\right\} \quad (42)$$

in such a way that energy levels of the coupled system are given by $E_{jnm}^{(\xi)} = E_{nm} + \mathcal{E}_{jnm}^{(\xi)}$ with $\mathcal{E}_{jnm}^{(\xi)}$ obtained with (37) and (41) in the $\xi = \mathrm{U}$ and $\xi = \mathrm{N}$ interaction cases, respectively.

In concluding this section, we observe that by using relations (8) it is possible to obtain the excited states of the coupled system from a *background* state by using the expression $\left|\Psi_{jnm}^{(\xi)}\right\rangle = \hat{K}_{jnm}^{(\xi)}|\psi_{00}\rangle$ with the raising operator $\hat{K}_{jnm}^{(\xi)}$ and the background state $|\psi_{00}\rangle$ given by

$$\begin{aligned} \hat{K}_{jnm}^{(\xi)} &= \hat{\mathcal{T}}_{\Xi}\sum_{i=1}^{3} C_{ijnm}^{(\xi)}\left\{\hat{K}_{+}^{(1)}\right\}^n\left\{\hat{K}_{+}^{(2)}\right\}^m\hat{\sigma}_{ii}, \\ |\psi_{00}\rangle &= |0\rangle_1|0\rangle_2|1\rangle_{\mathrm{A}} + |0\rangle_1|1\rangle_2|2\rangle_{\mathrm{A}} + |1\rangle_1|1\rangle_2|3\rangle_{\mathrm{A}} \end{aligned} \quad (43)$$

and the single potential raising operator defined as

$$\hat{K}_{+}^{(k)} = \frac{1}{\sqrt{\hat{\mathcal{H}}_{-}^{(k)}}}\hat{B}_{+}^{(k)} \qquad \text{with} \qquad k = 1, 2. \quad (44)$$

3.3. The parasupersymmetric model for Λ configuration

3.3.1. The coupling potentials and the interaction Hamiltonians. Within the set of assumptions presented previously we introduce the Hamiltonian $\hat{H}_\Lambda = \hat{H}_{\mathrm{P}}^{(\Lambda)} + \hat{H}_{\xi}^{(\Lambda)}$ which describe the coupling of a three-level system with two shape-invariant potentials in a Λ-type of configuration (see figure 1) where the parasupersymmetric part $\hat{H}_{\mathrm{P}}^{(\Lambda)}$ is given by

$$\hat{H}_{\mathrm{P}}^{(\Lambda)} = \hbar\Omega\big\{\big(\hat{A}_1\hat{A}_1^\dagger + \hat{A}_2^\dagger\hat{A}_2\big)\hat{\sigma}_{11} + \big(\hat{A}_1\hat{A}_1^\dagger + \hat{A}_2\hat{A}_2^\dagger\big)\hat{\sigma}_{22} + \big(\hat{A}_1^\dagger\hat{A}_1 + \hat{A}_2\hat{A}_2^\dagger\big)\hat{\sigma}_{33}\big\}, \tag{45}$$

while the interaction Hamiltonian for the two cases that we are considering is assumed with the forms

$$\hat{H}_{\mathrm{U}}^{(\Lambda)} = \hbar g\big\{\big(\hat{A}_1\hat{\sigma}_{23} + \hat{A}_1^\dagger\hat{\sigma}_{32}\big) + \big(\hat{A}_2^\dagger\hat{\sigma}_{12} + \hat{A}_2\hat{\sigma}_{21}\big)\big\}. \tag{46}$$

$$\hat{H}_{\mathrm{N}}^{(\Lambda)} = \hbar g\big\{\big(\hat{A}_1\sqrt{\hat{N}_1}\hat{\sigma}_{23} + \sqrt{\hat{N}_1}\hat{A}_1^\dagger\hat{\sigma}_{32}\big) + \big(\sqrt{\hat{N}_2}\hat{A}_2^\dagger\hat{\sigma}_{12} + \hat{A}_2\sqrt{\hat{N}_2}\hat{\sigma}_{21}\big)\big\}. \tag{47}$$

The algebraic formulation for shape-invariant systems when applied makes possible to write the Hamiltonian $\hat{H}_\Lambda$ in the form $\hat{H}_\Lambda = \hat{\mathcal{T}}_\Lambda\hat{h}_\Lambda\hat{\mathcal{T}}_\Lambda^\dagger$ if we define the parameter translation inclusive operator $\hat{\mathcal{T}}_\Lambda = \hat{T}_1\hat{\sigma}_{11} + \hat{T}_1\hat{T}_2\hat{\sigma}_{22} + \hat{T}_2\hat{\sigma}_{33}$ and decompose the Hamiltonian $\hat{h}_\Lambda$ in $\hat{h}_\Lambda = \hat{h}_{\mathrm{P}}^{(\Lambda)} + \hat{h}_{\xi}^{(\Lambda)}$, where the correspondent parasupersymmetric part is

$$\hat{h}_{\mathrm{P}}^{(\Lambda)} = \hbar\Omega\big\{\big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_-^{(2)}\big)\hat{\sigma}_{11} + \big(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_+^{(2)}\big)\hat{\sigma}_{22} + \big(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_+^{(2)}\big)\hat{\sigma}_{33}\big\} \tag{48}$$

and the interaction part $\hat{h}_{\xi}^{(\Lambda)}$ in the two cases is given by

$$\hat{h}_{\mathrm{U}}^{(\Lambda)} = \hbar g\big(\hat{B}_-^{(1)}\hat{\sigma}_{23} + \hat{B}_+^{(1)}\hat{\sigma}_{32} + \hat{B}_+^{(2)}\hat{\sigma}_{12} + \hat{B}_-^{(2)}\hat{\sigma}_{21}\big), \tag{49}$$

$$\hat{h}_{\mathrm{N}}^{(\Lambda)} = \hbar g\Big(\hat{B}_-^{(1)}\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{\sigma}_{23} + \sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{B}_+^{(1)}\hat{\sigma}_{32} + \sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{B}_+^{(2)}\hat{\sigma}_{12} + \hat{B}_-^{(2)}\sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{\sigma}_{21}\Big). \tag{50}$$

3.3.2. Parasupersymmetric algebra. Using the definition of the parasupercharge operator $\hat{\mathcal{Q}}_1$ presented in (25) and redefining the other two as

$$\hat{\mathcal{Q}}_2 = \hat{B}_+^{(2)}\hat{\sigma}_{12}, \qquad \hat{\mathcal{Q}}_2^\dagger = \hat{B}_-^{(2)}\hat{\sigma}_{21}$$

and

$$\hat{\mathcal{Q}}_{12} = \hat{B}_-^{(1)}\hat{\sigma}_{13} + \hat{B}_-^{(2)}\hat{\sigma}_{31}, \qquad \hat{\mathcal{Q}}_{12}^\dagger = \hat{B}_+^{(1)}\hat{\sigma}_{31} + \hat{B}_+^{(2)}\hat{\sigma}_{13} \tag{51}$$

it is possible to write the Hamiltonian $\hat{h}_{\mathrm{P}}^{(\Lambda)}$ in (48) with the same form (26) in terms of the parasupercharge anticommutators. Moreover, it is interesting to point out that the set of relations (27), (28) and (29) which characterizes the two-dimensional generalization of the second-order parasupersymmetric algebra remains valid. On the other hand, with this partial redefinition in the parasupercharge operators the Hamiltonian $\hat{h}_{\xi}^{(\Lambda)}$ for $\xi = \mathrm{U}$ and $\xi = \mathrm{N}$ interaction cases remains with the forms (30) and (31) since we keep the previous $\hat{\mathcal{P}}_1$ operator definition and change the $\hat{\mathcal{P}}_2$ operator definition by $\hat{\mathcal{P}}_2 = \sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{\mathcal{Q}}_2$. In both cases the commutation relations are observed $\big[\hat{\mathcal{Q}}_k, \hat{h}_{\mathrm{P}}^{(\Lambda)}\big] = \big[\hat{\mathcal{Q}}_k^\dagger, \hat{h}_{\mathrm{P}}^{(\Lambda)}\big] = \big[\hat{\mathcal{P}}_k, \hat{h}_{\mathrm{P}}^{(\Lambda)}\big] = \big[\hat{\mathcal{P}}_k^\dagger, \hat{h}_{\mathrm{P}}^{(\Lambda)}\big] = 0$ implying as a consequence that $\big[\hat{h}_{\mathrm{P}}^{(\Lambda)}, \hat{h}_{\xi}^{(\Lambda)}\big] = 0$ and thus $\hat{H}_{\mathrm{P}}^{(\Lambda)}$ and $\hat{H}_{\xi}^{(\Lambda)}$ share a common set of eigenstates. The action of the operators $\hat{\mathcal{Q}}_k$ or $\hat{\mathcal{P}}_k$ and $\hat{\mathcal{Q}}_k^\dagger$ or $\hat{\mathcal{P}}_k^\dagger$ in the heating and cooling process of the coupled system is illustrated in figure 1.

3.3.3. Eigenstates and eigenvalues. In this case, the use of the state (32) with the parameter translation intrinsic operator $\hat{\mathcal{T}}_\Lambda$ and (49) in the eigenstates equation $\hat{H}_{\rm U}^{(\Lambda)}\big|\Psi_\alpha^{(\rm U)}\big\rangle = \mathcal{E}_\alpha^{(\rm U)}\big|\Psi_\alpha^{(\rm U)}\big\rangle$ results in the system of equations

$$\begin{cases} \hbar g\sqrt{e_{m_2+1}^{(2)}}\big(\hat{T}_1\hat{T}_2C_{2\alpha}^{(\rm U)}\hat{T}_2^\dagger\hat{T}_1^\dagger\big)\hat{T}_1|n_2\rangle_1|m_2+1\rangle_2 = \mathcal{E}_\alpha^{(\rm U)}\big(\hat{T}_1C_{1\alpha}^{(\rm U)}\hat{T}_1^\dagger\big)\hat{T}_1|n_1\rangle_1|m_1\rangle_2 \\ \hbar g\big\{\sqrt{e_{m_1}^{(2)}}\big(\hat{T}_1C_{1\alpha}^{(\rm U)}\hat{T}_1^\dagger\big)\hat{T}_1\hat{T}_2|n_1\rangle_1|m_1-1\rangle_2 + \sqrt{e_{n_3}^{(1)}}\big(\hat{T}_2C_{3\alpha}^{(\rm U)}\hat{T}_2^\dagger\big)\hat{T}_1\hat{T}_2|n_3-1\rangle_1|m_3\rangle_2\big\} \\ \quad = \mathcal{E}_\alpha^{(\rm U)}\big(\hat{T}_1\hat{T}_2C_{2\alpha}^{(\rm U)}\hat{T}_2^\dagger\hat{T}_1^\dagger\big)\hat{T}_1\hat{T}_2|n_2\rangle_1|m_2\rangle_2 \\ \hbar g\sqrt{e_{n_2+1}^{(1)}}\big(\hat{T}_1\hat{T}_2C_{2\alpha}^{(\rm U)}\hat{T}_2^\dagger\hat{T}_1^\dagger\big)\hat{T}_2|n_2+1\rangle_1|m_2\rangle_2 = \mathcal{E}_\alpha^{(\rm U)}\big(\hat{T}_2C_{3\alpha}^{(\rm U)}\hat{T}_2^\dagger\big)\hat{T}_2|n_3\rangle_1|m_3\rangle_2 \end{cases} \tag{52}$$

the solution of which, obtained following the same steps of the previous case of the Ξ-type of configuration, gives the same three eigenvalues $\mathcal{E}_{jnm}^{(\rm U)}$ of equation (37) and the associated eigenstates

$$\big|\Psi_{jnm}^{(\rm U)}\big\rangle = \hat{\mathcal{T}}_\Lambda\big\{C_{1jnm}^{(\rm U)}|n\rangle_1|m+1\rangle_2|1\rangle_{\rm A} + C_{2jnm}^{(\rm U)}|n\rangle_1|m\rangle_2|2\rangle_{\rm A} + C_{3jnm}^{(\rm U)}|n+1\rangle_1|m\rangle_2|3\rangle_{\rm A}\big\}, \tag{53}$$

with the expansion coefficients satisfying the normalization condition (39) and the additional condition

$$\hbar g\big(\hat{T}_1\hat{T}_2C_{2jnm}^{(\rm U)}\hat{T}_1^\dagger\hat{T}_2^\dagger\big) = \frac{\mathcal{E}_{jnm}^{(\rm U)}}{\sqrt{e_{m+1}^{(2)}}}\big(\hat{T}_1C_{1jnm}^{(\rm U)}\hat{T}_1^\dagger\big) = \frac{\mathcal{E}_{jnm}^{(\rm U)}}{\sqrt{e_{n+1}^{(1)}}}\big(\hat{T}_2C_{3jnm}^{(\rm U)}\hat{T}_2^\dagger\big). \tag{54}$$

It is not difficult to show that the eigenvalue equation $\hat{H}_{\rm N}^{(\Lambda)}\big|\Psi_\alpha^{(\rm N)}\big\rangle = \mathcal{E}_\alpha^{(\rm N)}\big|\Psi_\alpha^{(\rm N)}\big\rangle$ for $\xi = {\rm N}$ interaction case must be the same eigenvalues $\mathcal{E}_{jnm}^{(\rm N)}$ obtained in (41) for the Ξ-type of configuration and the associated eigenstates with the form (53) but with the change $\sqrt{e_\nu^{(k)}} \to e_\nu^{(k)}$ of the coupling potential eigenvalue factors in the expansion coefficient $C_{ijnm}^{(\rm N)}$ condition corresponding to (54). One can easily check that the eigenstates $\big|\Psi_{jnm}^{(\rm U)}\big\rangle$ and $\big|\Psi_{jnm}^{(\rm N)}\big\rangle$ for the two kind of the interaction Hamiltonians $\hat{H}_{\rm U}^{(\Lambda)}$ and $\hat{H}_{\rm N}^{(\Lambda)}$ are eigenstates of the Hamiltonian $\hat{H}_{\rm P}^{(\Lambda)}$ of (45) with the same eigenvalues E_{nm} given by equation (42).

To conclude this section, we observe that the excited states of the coupled system $\big|\Psi_{jnm}^{(\xi)}\big\rangle$ can be obtained from the background state $|\psi_{00}\rangle = |0\rangle_1|1\rangle_2|1\rangle_{\rm A} + |0\rangle_1|0\rangle_2|2\rangle_{\rm A} + |1\rangle_1|0\rangle_2|3\rangle_{\rm A}$ by using the expression $\big|\Psi_{jnm}^{(\xi)}\big\rangle = \hat{K}_{jnm}^{(\xi)}|\psi_{00}\rangle$ with the raising operator obtained by equation (43) and the parameter translation inclusive operator given by $\hat{\mathcal{T}}_\Lambda$.

3.4. The parasupersymmetric model for V *configuration*

3.4.1. The coupling potentials and the interaction Hamiltonians. The coupling of a three-level system with two shape-invariant potentials in a V-type of configuration (see figure 1) has its parasupersymmetric Hamiltonian part given by

$$\hat{H}_{\rm P}^{(\rm V)} = \hbar\Omega\big\{\big(\hat{A}_1^\dagger\hat{A}_1 + \hat{A}_2\hat{A}_2^\dagger\big)\hat{\sigma}_{11} + \big(\hat{A}_1^\dagger\hat{A}_1 + \hat{A}_2^\dagger\hat{A}_2\big)\hat{\sigma}_{22} + \big(\hat{A}_1\hat{A}_1^\dagger + \hat{A}_2^\dagger\hat{A}_2\big)\hat{\sigma}_{33}\big\}, \tag{55}$$

while the interaction Hamiltonian for the two cases that we are considering now is assumed with the forms

$$\hat{H}_{\rm U}^{(\rm V)} = \hbar g\big\{\big(\hat{A}_1^\dagger\hat{\sigma}_{23} + \hat{A}_1\hat{\sigma}_{32}\big) + \big(\hat{A}_2\hat{\sigma}_{12} + \hat{A}_2^\dagger\hat{\sigma}_{21}\big)\big\} \tag{56}$$

$$\hat{H}_{\rm N}^{(\rm V)} = \hbar g\big\{\big(\sqrt{\hat{N}_1}\hat{A}_1^\dagger\hat{\sigma}_{23} + \hat{A}_1\sqrt{\hat{N}_1}\hat{\sigma}_{32}\big) + \big(\hat{A}_2\sqrt{\hat{N}_2}\hat{\sigma}_{12} + \sqrt{\hat{N}_2}\hat{A}_2^\dagger\hat{\sigma}_{21}\big)\big\}. \tag{57}$$

When applied in the Hamiltonian $\hat{H}_{\rm V} = \hat{H}_{\rm P}^{(\rm V)} + \hat{H}_\xi^{(\rm V)}$, the algebraic formulation of section 2 for shape-invariant systems gives $\hat{H}_{\rm V} = \hat{\mathcal{T}}_{\rm V}\hat{h}_{\rm V}\hat{\mathcal{T}}_{\rm V}^\dagger$ if we define the parameter translation inclusive operator $\hat{\mathcal{T}}_{\rm V} = \hat{T}_2\hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{T}_1\hat{\sigma}_{33}$ and decompose the Hamiltonian $\hat{h}_{\rm V}$ in $\hat{h}_{\rm V} = \hat{h}_{\rm P}^{(\rm V)} + \hat{h}_{\xi}^{(\rm V)}$ where

$$\hat{h}_{\rm P}^{(\rm V)} = \hbar\Omega\left\{\left(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_+^{(2)}\right)\hat{\sigma}_{11} + \left(\hat{\mathcal{H}}_-^{(1)} + \hat{\mathcal{H}}_-^{(2)}\right)\hat{\sigma}_{22} + \left(\hat{\mathcal{H}}_+^{(1)} + \hat{\mathcal{H}}_-^{(2)}\right)\hat{\sigma}_{33}\right\} \tag{58}$$

and the interaction part, in the two forms considered, is given by

$$\hat{h}_{\rm U}^{(\rm V)} = \hbar g\left(\hat{B}_+^{(1)}\hat{\sigma}_{23} + \hat{B}_-^{(1)}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\hat{\sigma}_{12} + \hat{B}_+^{(2)}\hat{\sigma}_{21}\right), \tag{59}$$

$$\hat{h}_{\rm N}^{(\rm V)} = \hbar g\left(\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{B}_+^{(1)}\hat{\sigma}_{23} + \hat{B}_-^{(1)}\sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{\sigma}_{32} + \hat{B}_-^{(2)}\sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{\sigma}_{12} + \sqrt{\hat{\mathcal{H}}_-^{(2)}}\hat{B}_+^{(2)}\hat{\sigma}_{21}\right). \tag{60}$$

3.4.2. Parasupersymmetry algebra. Keeping the definition of the parasupercharge operator $\hat{\mathcal{Q}}_2$ presented in (25) and redefining the other two as

$$\hat{\mathcal{Q}}_1 = \hat{B}_+^{(1)}\hat{\sigma}_{23}, \qquad \hat{\mathcal{Q}}_1^\dagger = \hat{B}_-^{(1)}\hat{\sigma}_{32}$$

and

$$\hat{\mathcal{Q}}_{12} = \hat{B}_+^{(1)}\hat{\sigma}_{13} + \hat{B}_+^{(2)}\hat{\sigma}_{31}, \qquad \hat{\mathcal{Q}}_{12}^\dagger = \hat{B}_-^{(1)}\hat{\sigma}_{31} + \hat{B}_-^{(2)}\hat{\sigma}_{13}, \tag{61}$$

the Hamiltonian $\hat{h}_{\rm P}^{(\rm V)}$ in (58) can be written with the same form (26) in terms of anticommutators and the set of parasupersymmetric algebraic relations (27), (28) and (29) remains valid. The partial redefinition in the supercharge operators assures that the interaction Hamiltonians $\hat{h}_{\rm U}^{(\rm V)}$ and $\hat{h}_{\rm N}^{(\rm V)}$ keep the forms (30) and (31) with $\hat{\mathcal{P}}_1 = \sqrt{\hat{\mathcal{H}}_-^{(1)}}\hat{\mathcal{Q}}_1$ and $\hat{\mathcal{P}}_2 = \hat{\mathcal{Q}}_2\sqrt{\hat{\mathcal{H}}_-^{(2)}}$. For both interaction cases, the Hamiltonians $\hat{H}_{\rm P}^{(\rm V)}$ and $\hat{H}_{\xi}^{(\rm V)}$ share a common set of eigenstates since $\left[\hat{\mathcal{Q}}_k, \hat{h}_{\rm P}^{(\rm V)}\right] = \left[\hat{\mathcal{Q}}_k^\dagger, \hat{h}_{\rm P}^{(\rm V)}\right] = \left[\hat{\mathcal{P}}_k, \hat{h}_{\rm P}^{(\rm V)}\right] = \left[\hat{\mathcal{P}}_k^\dagger, \hat{h}_{\rm P}^{(\rm V)}\right] = 0$ which assure that $\left[\hat{h}_{\rm P}^{(\rm V)}, \hat{h}_{\xi}^{(\rm V)}\right] = 0$.

3.4.3. Eigenstates and eigenvalues. With the dressed state (32), the parameter translation operator $\hat{\mathcal{T}}_{\rm V}$ and (59) in the eigenstates equation $\hat{H}_{\rm U}^{(\rm V)}\left|\Psi_\alpha^{(\rm U)}\right\rangle = \mathcal{E}_\alpha^{(\rm U)}\left|\Psi_\alpha^{(\rm U)}\right\rangle$ we obtain the system of equations

$$\begin{cases} \hbar g\sqrt{e_{m_2}^{(2)}}C_{2\alpha}^{(\rm U)}\hat{T}_2|n_2\rangle_1|m_2-1\rangle_2 = \mathcal{E}_\alpha^{(\rm U)}\left(\hat{T}_2C_{1\alpha}^{(\rm U)}\hat{T}_2^\dagger\right)\hat{T}_2|n_1\rangle_1|m_1\rangle_2 \\ \hbar g\left\{\sqrt{e_{m_1+1}^{(2)}}\left(\hat{T}_2C_{1\alpha}^{(\rm U)}\hat{T}_2^\dagger\right)|n_1\rangle_1|m_1+1\rangle_2 + \sqrt{e_{n_3+1}^{(1)}}\left(\hat{T}_1C_{3\alpha}^{(\rm U)}\hat{T}_1^\dagger\right)|n_3+1\rangle_1|m_3\rangle_2\right\} \\ \quad = \mathcal{E}_\alpha^{(\rm U)}C_{2\alpha}^{(\rm U)}|n_2\rangle_1|m_2\rangle_2 \\ \hbar g\sqrt{e_{n_2}^{(1)}}C_{2\alpha}^{(\rm U)}\hat{T}_1|n_2-1\rangle_1|m_2\rangle_2 = \mathcal{E}_\alpha^{(\rm U)}\left(\hat{T}_1C_{3\alpha}^{(\rm U)}\hat{T}_1^\dagger\right)\hat{T}_1|n_3\rangle_1|m_3\rangle_2 \end{cases} \tag{62}$$

the solution of which, obtained following the same steps of the previous level configuration cases, gives the same three eigenvalues $\mathcal{E}_{jnm}^{(\rm U)}$ of equation (37) and the associated eigenstates

$$\left|\Psi_{jnm}^{(\rm U)}\right\rangle = \hat{\mathcal{T}}_{\rm V}\left\{C_{1jnm}^{(\rm U)}|n+1\rangle_1|m\rangle_2|1\rangle_{\rm A} + C_{2jnm}^{(\rm U)}|n+1\rangle_1|m+1\rangle_2|2\rangle_{\rm A} + C_{3jnm}^{(\rm U)}|n\rangle_1|m+1\rangle_2|3\rangle_{\rm A}\right\}, \tag{63}$$

with the expansion coefficients satisfying the normalization condition (39) and the additional condition

$$\hbar g C_{2jnm}^{(\rm U)} = \frac{\mathcal{E}_{jnm}^{(\rm U)}}{\sqrt{e_{m+1}^{(2)}}}\left(\hat{T}_2C_{1jnm}^{(\rm U)}\hat{T}_2^\dagger\right) = \frac{\mathcal{E}_{jnm}^{(\rm U)}}{\sqrt{e_{n+1}^{(1)}}}\left(\hat{T}_1C_{3jnm}^{(\rm U)}\hat{T}_1^\dagger\right). \tag{64}$$

As in the two previous configuration types, we find the same eigenvalues $\mathcal{E}_{jnm}^{(\rm N)}$ given by equation (41) for the eigenvalue equation $\hat{H}_{\rm N}^{(\rm V)}\left|\Psi_\alpha^{(\rm N)}\right\rangle = \mathcal{E}_\alpha^{(\rm N)}\left|\Psi_\alpha^{(\rm N)}\right\rangle$ in the $\xi = \rm N$ interaction case. However, the associated eigenstates with the form (63) have its expansion coefficients $C_{ijnm}^{(\mathrm{N})}$ satisfying similar conditions to (64) replacing the coupling potential eigenvalue factors $\sqrt{e_{\nu}^{(k)}} \to e_{\nu}^{(k)}$. It is trivial to verify that the eigenstates $\left|\Psi_{jnm}^{(\mathrm{U})}\right\rangle$ and $\left|\Psi_{jnm}^{(\mathrm{N})}\right\rangle$ for the two kinds of the interaction Hamiltonians $\hat{H}_{\xi}^{(\mathrm{V})}$ also are eigenstates of the Hamiltonian $\hat{H}_{\mathrm{P}}^{(\mathrm{V})}$ of (55) with the same set of eigenvalues E_{nm} given by equation (42) and obtained in the other two types of configuration.

To close this section, we observe that with the background state $|\psi_{00}\rangle = |1\rangle_1|0\rangle_2|1\rangle_{\mathrm{A}} + |1\rangle_1|1\rangle_2|2\rangle_{\mathrm{A}} + |0\rangle_1|1\rangle_2|3\rangle_{\mathrm{A}}$ it is possible to obtain the excited states of the coupled system $\left|\Psi_{jnm}^{(\xi)}\right\rangle$ by using the expression $\left|\Psi_{jnm}^{(\xi)}\right\rangle = \hat{K}_{jnm}^{(\xi)}|\psi_{00}\rangle$ with the raising operator obtained by equation (43) and the parameter translation inclusive operator given by $\hat{\mathcal{T}}_{\mathrm{V}}$.

4. Application for a couple of potentials

To illustrate how our general results can be applied in specific cases we work out in this section an objective example of a three-level system coupled with a harmonic oscillator and a Morse potentials. The harmonic oscillator is the simplest among the shape-invariant potential and appears in the description of the interaction of matter, represented by a few level atom, with a quantized electromagnetic field, represented by the harmonic oscillator bosonic operators $\hat{A}_1\big(a_1^{(1)}\big)$ and $\hat{A}_1^{\dagger}\big(a_1^{(1)}\big)$. On the other hand, one-dimensional Morse potential, originally introduced as a useful model for the diatomic molecules [34], has been widely used in many areas of physics to study physics phenomena such as molecular vibrations, laser chemistry and, in particular, chemical bonds. With the inclusion of the Morse potential it is possible to evaluate the anharmonic and dissociation effects, related to a more realistic physical situation, in the coupled system.

In the case of the harmonic oscillator the partner potentials (3) are obtained with the superpotential $W_1\big(x, a_1^{(1)}\big) = \sqrt{\hbar\Omega}\big(a_1^{(1)}x + \zeta\big)$, where $a_1^{(1)}$ and ζ are real constants, while the remainders [4] in the shape-invariant condition (5) are given by $R_1\big(a_n^{(1)}\big) = \eta_1\big(a_n^{(1)} + a_{n+1}^{(1)}\big)$, where $\eta_1 = \sqrt{\hbar/(2M\Omega)}$. Taking into account that the parameters for this potential are related by $a_1^{(1)} = a_2^{(1)} = \cdots = a_n^{(1)}$ then the remainders can be written as $R_1\big(a_n^{(1)}\big) = \gamma$, with $\gamma = 2\eta_1 a_1^{(1)}$, and thus

$$e_n^{(1)} = n\gamma. \tag{65}$$

The supersymmetric partner potentials (3) in the Morse case [4] are obtained with the superpotential $W_2\big(y, a_1^{(2)}\big) = \sqrt{\hbar\Omega}\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y}\big\}$, with $a_1^{(2)}$ and ϱ being real constants. In this case the remainders [4] in the shape-invariant condition (5) are given by $R_2\big(a_m^{(2)}\big) = \eta_2\big(2a_m^{(2)} - \eta_2\big)$, with the potential parameters related by $a_{m+1}^{(2)} = a_m^{(2)} - \eta_2$, where $\eta_2 = \sqrt{\hbar/(2M\Omega)}\varrho$. By using these results into (9) we can prove that the bound states are related to

$$e_m^{(2)} = \eta_2^2 m(2\kappa - m), \qquad \text{with} \qquad m \leqslant \kappa \equiv a_1^{(2)}\big/\eta_2. \tag{66}$$

With this couple of potentials the parasupersymmetric Hamiltonians $\hat{H}_{\mathrm{P}}$ in (19), (45), (55) and the interactions Hamiltonians $\hat{H}_{\xi}^{(\mathrm{X})}$ in (20), (21), (46), (47), (56) and (57) for the three types of configuration of the three level atom must be constructed with the dimensionless operators $\hat{A}_1\big(a_1^{(1)}, x\big) = a_1^{(1)}x + \zeta + \mathrm{i}\lambda\hat{p}_x$, $\hat{A}_1^{\dagger}\big(a_1^{(1)}, x\big) = a_1^{(1)}x + \zeta - \mathrm{i}\lambda\hat{p}_x$, $\hat{A}_2\big(a_1^{(2)}, y\big) = a_1^{(2)} - \mathrm{e}^{-\varrho y} + \mathrm{i}\lambda\hat{p}_y$ and $\hat{A}_2^{\dagger}\big(a_1^{(2)}, y\big) = a_1^{(2)} - \mathrm{e}^{-\varrho y} - \mathrm{i}\lambda\hat{p}_y$, where $\lambda = 1/\sqrt{2\hbar\Omega M}$. In a matrix form the

Hamiltonian $\hat{H}_{\mathrm{X}} = \hat{H}_{\mathrm{P}}^{(\mathrm{X})} + \hat{H}_{\xi}^{(\mathrm{X})}$ for the usual ($\xi = \mathrm{U}$) interaction case and the three types of configuration gives

$$\hat{H}_{\Xi} = \hbar\Omega \begin{bmatrix} \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} + \mathrm{i}\lambda\hat{p}_y\big\} & 0 \\ \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} - \mathrm{i}\lambda\hat{p}_y\big\} & \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(1)}x + \zeta + \mathrm{i}\lambda\hat{p}_x\big\} \\ 0 & \varepsilon\big\{a_1^{(1)}x + \zeta - \mathrm{i}\lambda\hat{p}_x\big\} & \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) \end{bmatrix} \tag{67}$$

$$\hat{H}_{\Lambda} = \hbar\Omega \begin{bmatrix} \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} - \mathrm{i}\lambda\hat{p}_y\big\} & 0 \\ \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} + \mathrm{i}\lambda\hat{p}_y\big\} & \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(1)}x + \zeta + \mathrm{i}\lambda\hat{p}_x\big\} \\ 0 & \varepsilon\big\{a_1^{(1)}x + \zeta - \mathrm{i}\lambda\hat{p}_x\big\} & \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) \end{bmatrix} \tag{68}$$

$$\hat{H}_{\mathrm{V}} = \hbar\Omega \begin{bmatrix} \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} + \mathrm{i}\lambda\hat{p}_y\big\} & 0 \\ \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} - \mathrm{i}\lambda\hat{p}_y\big\} & \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(1)}x + \zeta - \mathrm{i}\lambda\hat{p}_x\big\} \\ 0 & \varepsilon\big\{a_1^{(1)}x + \zeta + \mathrm{i}\lambda\hat{p}_x\big\} & \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) \end{bmatrix}, \tag{69}$$

where $\varepsilon = g/\Omega$ and $\hat{h}_{\mathrm{o}} = \lambda^2\big(\hat{p}_x^2 + \hat{p}_y^2\big) + \big(a_1^{(1)}x + \zeta\big)^2 + \big(a_1^{(2)} - \mathrm{e}^{-\varrho y}\big)^2$. For the nonlinear ($\xi = \mathrm{N}$) interaction case, these three Hamiltonians have the matrix form

$$\hat{H}_{\Xi} = \hbar\Omega \begin{bmatrix} \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} + \mathrm{i}\lambda\hat{p}_y\big\}\sqrt{\hat{h}_2} & 0 \\ \varepsilon\sqrt{\hat{h}_2}\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} - \mathrm{i}\lambda\hat{p}_y\big\} & \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(1)}x + \zeta + \mathrm{i}\lambda\hat{p}_x\big\}\sqrt{\hat{h}_1} \\ 0 & \varepsilon\sqrt{\hat{h}_1}\big\{a_1^{(1)}x + \zeta - \mathrm{i}\lambda\hat{p}_x\big\} & \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) \end{bmatrix} \tag{70}$$

$$\hat{H}_{\Lambda} = \hbar\Omega \begin{bmatrix} \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\sqrt{\hat{h}_2}\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} - \mathrm{i}\lambda\hat{p}_y\big\} & 0 \\ \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} + \mathrm{i}\lambda\hat{p}_y\big\}\sqrt{\hat{h}_2} & \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(1)}x + \zeta + \mathrm{i}\lambda\hat{p}_x\big\}\sqrt{\hat{h}_1} \\ 0 & \varepsilon\sqrt{\hat{h}_1}\big\{a_1^{(1)}x + \zeta - \mathrm{i}\lambda\hat{p}_x\big\} & \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) \end{bmatrix} \tag{71}$$

$$\hat{H}_{\mathrm{V}} = \hbar\Omega \begin{bmatrix} \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} + \mathrm{i}\lambda\hat{p}_y\big\}\sqrt{\hat{h}_2} & 0 \\ \varepsilon\sqrt{\hat{h}_2}\big\{a_1^{(2)} - \mathrm{e}^{-\varrho y} - \mathrm{i}\lambda\hat{p}_y\big\} & \hat{h}_{\mathrm{o}} - \hbar\lambda\big(a_1^{(1)} + \varrho\,\mathrm{e}^{-\varrho y}\big) & \varepsilon\sqrt{\hat{h}_1}\big\{a_1^{(1)}x + \zeta - \mathrm{i}\lambda\hat{p}_x\big\} \\ 0 & \varepsilon\big\{a_1^{(1)}x + \zeta + \mathrm{i}\lambda\hat{p}_x\big\}\sqrt{\hat{h}_1} & \hat{h}_{\mathrm{o}} + \hbar\lambda\big(a_1^{(1)} - \varrho\,\mathrm{e}^{-\varrho y}\big) \end{bmatrix}, \tag{72}$$

where $\hat{h}_1 = \lambda^2\hat{p}_x^2 + \big(a_1^{(1)}x + \zeta\big)^2 - \hbar\lambda a_1^{(1)}$ and $\hat{h}_2 = \lambda^2\hat{p}_y^2 + \big(a_1^{(2)} - \mathrm{e}^{-\varrho y}\big)^2 - \hbar\lambda\varrho\,\mathrm{e}^{-\varrho y}$.

The eigenvalues of the coupled system for the two kind of interactions are given by

$$E_{jnm}^{(\mathrm{U})} = \hbar\Omega\big\{\gamma(n+1) + \eta_2^2(m+1)(2\kappa - m - 1)\big\} \times \begin{cases} +\hbar g\sqrt{\gamma(n+1) + \eta_2^2(m+1)(2\kappa - m - 1)} & j = 1 \\ +0 & j = 2 \\ -\hbar g\sqrt{\gamma(n+1) + \eta_2^2(m+1)(2\kappa - m - 1)} & j = 3 \end{cases} \tag{73}$$

and

$$E^{(\mathrm{N})}_{jnm} = \hbar\Omega\big\{\gamma(n+1) + \eta_2^2(m+1)(2\kappa - m - 1)\big\} \times \begin{cases} +\hbar g\sqrt{\gamma^2(n+1)^2 + \eta_2^4(m+1)^2(2\kappa - m - 1)^2} & j = 1 \\ +0 & j = 2 \\ -\hbar g\sqrt{\gamma^2(n+1)^2 + \eta_2^4(m+1)^2(2\kappa - m - 1)^2} & j = 3, \end{cases} \tag{74}$$

with the correspondent eigenfunctions obtained by $\Psi^{(\mathrm{X},\xi)}_{jnm} \equiv \langle x, y | \Psi^{(\mathrm{X},\xi)}_{jnm}\rangle = \hat{\mathcal{T}}_{\mathrm{X}}\psi^{(\Xi,\xi)}_{jnm}$ with $\mathrm{X} = \Xi, \Lambda, \mathrm{V}$, where

$$\psi^{(\Xi,\xi)}_{jnm} = \begin{bmatrix} C^{(\Xi,\xi)}_{1jnm}\varphi_{n,m}(x, y) \\ C^{(\Xi,\xi)}_{2jnm}\varphi_{n,m+1}(x, y) \\ C^{(\Xi,\xi)}_{3jnm}\varphi_{n+1,m+1}(x, y) \end{bmatrix}, \qquad \psi^{(\Lambda,\xi)}_{jnm} = \begin{bmatrix} C^{(\Lambda,\xi)}_{1jnm}\varphi_{n+1,m}(x, y) \\ C^{(\Lambda,\xi)}_{2jnm}\varphi_{n+1,m+1}(x, y) \\ C^{(\Lambda,\xi)}_{3jnm}\varphi_{n,m+1}(x, y) \end{bmatrix},$$
$$\psi^{(\mathrm{V},\xi)}_{jnm} = \begin{bmatrix} C^{(\mathrm{V},\xi)}_{1jnm}\varphi_{n,m+1}(x, y) \\ C^{(\mathrm{V},\xi)}_{2jnm}\varphi_{n,m}(x, y) \\ C^{(\mathrm{V},\xi)}_{3jnm}\varphi_{n+1,m}(x, y) \end{bmatrix}. \tag{75}$$

The spinor elements $\varphi_{\mu,\nu}(x, y)$ of the eigenfunctions $\Psi^{(\mathrm{X},\xi)}_{jnm}$ are obtained by $\varphi_{\mu\nu}(x, y) = \varphi^{(1)}_{\mu}\big[a^{(1)}_1 x + \zeta\big]\varphi^{(2)}_{\nu}\big[2\kappa\,\mathrm{e}^{-\varrho y}\big]$, where

$$\varphi^{(1)}_{\mu}(u) = \mathrm{e}^{-u^2/2}\mathrm{H}_{\mu}(u) \qquad \text{while} \qquad \varphi^{(2)}_{\nu}(u) = \mathrm{e}^{-u/2}u^{\alpha/2}\mathrm{L}^{\alpha}_{\nu}(u), \quad \text{with} \quad \alpha = 2\kappa - 2\nu - 1 \tag{76}$$

where $\mathrm{H}_{\mu}(u)$ are the Hermite polynomials [35] and $\mathrm{L}^{\alpha}_{\nu}(u)$ are the associated Laguerre polynomials [35].

5. Conclusions

Exactly soluble and fully quantum-mechanical models are rare. In this paper we introduced, within a parasupersymmetric formulation, a class of bound-state problems which represents a three-level atom coupled with a two-dimensional shape-invariant potential system. This represents a non-trivial coupled-channels problem which may find applications in molecular, atomic and nuclear physics. Taking into account the three possible configurations of the three energy level of the atom and two forms of coupling interaction (usual and nonlinear) we discussed the parasupersymmetric algebra of the models. Using a parasupersymmetric formulation, we showed that the shape-invariant potentials Hamiltonian $\hat{H}^{(\mathrm{X})}_{\mathrm{P}}$ and the interaction Hamiltonian $\hat{H}^{(\mathrm{X})}_{\xi}$ are mutually commuting and thus we obtained the eigenvalues and the eigenstates of the coupled system. We applied our generalized results for the particular case of a couple of shape-invariant potentials (harmonic oscillator + Morse potentials).

Because of its relevance an extension the study of the quantum dynamics of the coupled models, strongly depends on the initial conditions of the system, will be presented in a forthcoming calculation.

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